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Remark on the uniqueness of a mild solution of the Boltzmann equation in the critical Besov space

Dedicated Professor Yoshinori Morimoto on the occasion of his sixtyfifth birthday

By

Takayoshi OGAWA * and Senjo SHIMIZU **

Abstract

We consider the initial value problem of the Boltzmann equation in the whole space $\mathbb{R}^n$. After reviewing the standard formulation for time dependent Boltzmann equation around the Maxwellian steady state, we show the uniqueness of a (weak) mild solution for the equation in the critical Besov spaces. The result is along the analysis due to Ukai-Yang [34] and the improved weight estimate in the velocity space using maximal regularity estimate.

§ 1. Introduction

We consider the initial value problem for the Boltzmann equation in $\mathbb{R}^n \times \mathbb{R}^n$:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = Q(f,f), & t > 0, (v,x) \in \mathbb{R}^n \times \mathbb{R}^n, \\ f(0,v,x) = f_0(v,x), & t = 0, (v,x) \in \mathbb{R}^n \times \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ denotes the distribution of gas particles at time $t \in \mathbb{R}_+$ with velocity $v \in \mathbb{R}^n$ and position $x \in \mathbb{R}^n$. The function $f_0 = f_0(v,x) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a given initial condition. The bilinear term $Q(f,g)$ represents the Boltzmann collision operator between the each gas particles and is defined by the following quadratic form with the incoming and outgoing velocity parameters (v, v_*), (v', v'_*) with a direction

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$\omega \in \mathbb{S}^{n-1}$ after collision: Let a non-negative function $q(\cdot,\cdot) : \mathbb{R}_+ \times \mathbb{S}^{n-1} \to \mathbb{R}$ be the collision kernel, then

$$Q(f,g) = \int_{\mathbb{R}^n} \int_{\mathbb{S}^{n-1}} q\big(|v-v_*|,\omega\big)\big(f(v'_*)g(v') - f(v_*)g(v)\big) d\omega dv_*,$$

where

$$v' = v - [(v-v_*)\cdot\omega]\omega, \quad v'_* = v_* + [(v-v_*)\cdot\omega]\omega. \tag{1.2}$$

By the definition (1.2), we see that

$$\begin{aligned} v + v_* &= v' + v'_*, \\ |v|^2 + |v_*|^2 &= |v'|^2 + |v'_*|^2, \\ |v' - v'_*|^2 &= |v - (v-v_*,\omega)\omega - v_* - (v-v_*,\omega)\omega|^2 \\ &= |v-v_*|^2. \end{aligned} \tag{1.3}$$

Throughout this paper, we assume that the collision kernel q is of the angular cutoff type. Namely we assume Grad's angular cutoff assumption;

$$q(|v-v_*|,\omega) \le C|v-v_*|^\gamma B_0(\theta) \tag{1.4}$$

for some $0 < \gamma < 1$ and $B_0(\theta)$ is a bounded non-negative function of θ, where $\cos\theta = \frac{v-v_*}{|v-v_*|}\cdot\omega$.

The solvability of the initial value problem (1.1) is known in a various setting: After Grad [13] introduced the angular cut-off collision kernel, Ukai [29], [32] proved the global existence of a solution for (1.1) under the specially periodic boundary condition. Ukai-Asano [33] proved the global existence and uniqueness of a strong solution around the Maxwell distribution of the velocity field. Nishida-Imai [23] considered the global behavior of solutions (see for a bounded domain case, Shizuta-Asano [27]). Ukai [31] also proved the well-posedness of the solution in the Sobolev class embedded into a bounded class such as H^s ($s > 2$) with the hard sphere model (see also [11]). Diperna-Lions [10] constructed the existence of large data weak solutions (so called renormalized solutions) given up the uniqueness. Alexandre-Villani [4] studied the renormalized solutions in the non-cutoff case. Desvillettes-Villani [9] showed the convergence of solutions to the equilibrium (global Maxwellian) under strong smoothness assumptions for the solutions. Guo [15]-[17], T.P.Liu-Yang-Yu [20], [21] proved the existence of a unique classical solution by energy method. Duan-S.Liu-Xu [12] proved the existence and uniqueness of a global solution under the cutoff assumption in the critical Chemin-Lerner space of Besov type (cf. [7]). Morimoto-Sakamoto [22] extended the result under the non-cutoff assumption also including a soft potential case (see for results of non-cutoff cases, [1], [2], [8], [19], [35]).

In general, the function space for velocity and spatial variables in constructing a solution is taken as $L^2\big(\mathbb{R}^3_v; H^s(\mathbb{R}^3_x)\big)$ $(s > \frac{3}{2})$, where $H^s(\mathbb{R}^n)$ denotes the Sobolev space defined by the Bessel potential $\|f\|_{H^s} \equiv \|\langle\xi\rangle^s \hat{f}\|_2$. Then the embedding

$$L^2\big(\mathbb{R}^3_v; H^s(\mathbb{R}^3_x)\big) \subset L^2\big(\mathbb{R}^3_v; L^\infty(\mathbb{R}^3_x)\big)$$

ensures that the suitable bi-linear estimate is obtained and the control of the collision term is possibly done. Along this strategy, Ukai-Yang [34] considered the equation in the space $L^2(\mathbb{R}^n_v\times\mathbb{R}^n_x)\cap L^\infty_{\langle v\rangle^\alpha}(\mathbb{R}^n_v\times\mathbb{R}^n_x)$, where $L^\infty_{\langle v\rangle^\alpha}$ stands for the bounded functions with a velocity weight $\langle v\rangle^\alpha = (1+|v|^2)^{\alpha/2}$ $(\alpha > 0)$ and they employed the integral equation around the Maxwellian steady state to show the existence of the strong solution in the space. They developed the spectral analysis in the space and showed that the evolution operator recovers the velocity weight

$$\nu(v) \equiv \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} q(|v-v_*|,\omega)\sqrt{\mu_{v_*}}\,d\omega dv_* \simeq (1+|v|)^\gamma \tag{1.5}$$

for some $\gamma > 0$ under the assumption $\alpha > \frac{n}{2}$ with the aid of the velocity smoothing effect.

On the other hand, the energy method based on L^2 based space is useful to consider the existence of the strong solution. Among others, Duan-Liu-Xu [12] and Morimoto-Sakamoto [22] introduced the critical Chemin-Lerner space of the Besov type,

$$\widetilde{L^2\big(\mathbb{R}^3_v}; B^{\frac{3}{2}}_{2,1}(\mathbb{R}^3_x)\big)$$

instead of the Bochner space of Lebesgue-Sobolev type $L^2\big(\mathbb{R}^n_v; H^s(\mathbb{R}^n_x)\big)$ for non-cutoff case. The Chemin-Lerner space is useful to control the spatial variable and the regularity of the function space in x variable reaches the critical setting. In particular, the velocity weight is only $\nu(v)$ and it is an improvement of the result appearing in [34].

Definition (The Besov spaces). Let $\{\phi_j\}_{j\in\mathbb{Z}}$ be the Littlewood-Paley dyadic decomposition of unity, namely $\hat{\phi}$ is the Fourier transform of a smooth radial function ϕ with $\widehat{\phi}(\xi) \geq 0$,

$$\operatorname{supp}\widehat{\phi} \subset \{\xi\in\mathbb{R}^n |\ 2^{-1} < |\xi| < 2\}$$

and

$$\sum_{j\in\mathbb{Z}} \widehat{\phi_j}(\xi) = 1, \quad \widehat{\phi_j}(\xi) = \widehat{\phi}(2^{-j}\xi)$$

for all $\xi \neq 0$. For $s\in\mathbb{R}$ and $1 \leq p,\sigma \leq \infty$, we define the homogeneous Besov space $\dot{B}^s_{p,\sigma}(\mathbb{R}^n)$ by

$$\dot{B}^s_{p,\sigma}(\mathbb{R}^n) = \Big\{ f\in\mathcal{S}^*; f = \sum_{k\in\mathbb{Z}} \phi_k * f \text{ in } \mathcal{S}^*, \|f\|_{\dot{B}^s_{p,\sigma}} < \infty \Big\}$$

with the norm

$$\|f\|_{\dot{B}^s_{p,\sigma}} \equiv \begin{cases} \Big(\sum_{j\in\mathbb{Z}} 2^{js\sigma} \|\phi_j * f\|_p^\sigma \Big)^{1/\sigma}, & 1 \le \sigma < \infty, \\ \sup_{j\in\mathbb{Z}} 2^{js} \|\phi_j * f\|_p, & \sigma = \infty. \end{cases}$$

We also introduce the inhomogeneous Besov space $B^s_{p,\sigma}(\mathbb{R}^n)$ as

$$B^s_{p,\sigma}(\mathbb{R}^n) = \{f \in \mathcal{S}^*; \|f\|_{B^s_{p,\sigma}} < \infty\}$$

with the norm

$$\|f\|_{B^s_{p,\sigma}} \equiv \begin{cases} \Big(\|\psi * f\|_p^\sigma + \sum_{j\ge 0} 2^{js\sigma} \|\phi_j * f\|_p^\sigma \Big)^{1/\sigma}, & 1 \le \sigma < \infty, \\ \|\psi * f\|_p + \sup_{j\ge 0} 2^{js} \|\phi_j * f\|_p, & \sigma = \infty, \end{cases}$$

where $\widehat{\psi} = \widehat{\psi}(\xi)$ is a smooth cut off function with

$$\hat{\psi}(\xi) + \sum_{j\ge 0} \hat{\phi}_j(\xi) \equiv 1$$

for all $\xi \in \mathbb{R}^n$ (cf. [6], [28]). We use the abbreviated notation for the norm as

$$\|f\|_{B^s_{p,\sigma}} \equiv \begin{cases} \Big(\sum_{j\in\bar{\mathbb{N}}} 2^{js\sigma} \|\phi_j * f\|_p^\sigma \Big)^{1/\sigma}, & 1 \le \sigma < \infty, \\ \sup_{j\in\bar{\mathbb{N}}} 2^{js} \|\phi_j * f\|_p, & \sigma = \infty, \end{cases}$$

where $\bar{\mathbb{N}} = \{-1, 0, 1, 2, \cdots\}$ and we understand as $\phi_{-1} = \psi$.
Definition (Weighted Lebesgue spaces). For $\alpha \in \mathbb{R}$, $1 \le p \le \infty$, we say $f \in L^p_\alpha(\mathbb{R}^n_v)$ if

$$\|f\|_{L^p_\alpha} \equiv \big\| \langle v \rangle^\alpha f(v) \big\|_{L^p(\mathbb{R}^n_v)} < \infty,$$

where $\langle v \rangle = (1 + |v|^2)^{1/2}$ for $v \in \mathbb{R}^n$.

Duan et al. [12] introduced the function space involving the critical Besov space with the spatial variable x, which is slightly narrower than the usual Bochner space on $B^{\frac{n}{2}}_{2,1}(\mathbb{R}^n)$. Hence it is worth when we consider the Bochner spaces such as the space which embedded into $L^\infty(\mathbb{R}^n_x)$. The following embedding is known for the critical Besov spaces: For any $1 \le p < q \le \infty$,

$$B^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_x) \subsetneq B^{\frac{n}{q}}_{q,1}(\mathbb{R}^n_x) \subsetneq L^\infty(\mathbb{R}^n_x). \tag{1.6}$$

Whilst, the function space that Duan et al. introduced is of the Chemin-Lerner type [7];

$$\widetilde{L^\infty(0,T};L^2_\alpha\big(\mathbb{R}^n_v;B^{\frac{n}{2}}_{2,1}(\mathbb{R}^n_x)\big)\big) \tag{1.7}$$

equipped with the norm

$$\|f\|_{\widetilde{L^\infty(0,T};L^2_\alpha(\mathbb{R}^n_v;B^{\frac{n}{2}}_{2,1}(\mathbb{R}^n_x)))} \equiv \sum_{j\ge 0} 2^{\frac{n}{2}j}\Big\|\big\|\|\phi_j * f\|_{L^2(\mathbb{R}^n_x)}\big\|_{L^2_\alpha(\mathbb{R}^n_v)}\Big\|_{L^\infty(0,T)}.$$

The above function space is weaker than the space of the Bochner type :

$$L^\infty\Big(0,T;L^2_\alpha\big(\mathbb{R}^n_v;B^{\frac{n}{2}}_{2,1}(\mathbb{R}^n_x)\big)\Big),$$

equipped with the norm.

$$\|f\|_{L^\infty(0,T;L^2_\alpha(\mathbb{R}^n_v;B^{\frac{n}{2}}_{2,1}(\mathbb{R}^n_x)))} \equiv \left\|\Big\|\sum_{j\ge 0} 2^{\frac{n}{2}j}\|\phi_j * f\|_{L^2(\mathbb{R}^n_x)}\Big\|_{L^2_\alpha(\mathbb{R}^n_v)}\right\|_{L^\infty(0,T)}.$$

Namely

$$\widetilde{L^2_\alpha\big(\mathbb{R}^n_v};B^{\frac{n}{2}}_{2,1}(\mathbb{R}^n_x)\big) \subsetneq L^2_\alpha\big(\mathbb{R}^n_v;B^{\frac{n}{2}}_{2,1}(\mathbb{R}^n_x)\big),$$

and

$$\widetilde{L^\infty(0,T};L^2_\alpha\big(\mathbb{R}^n_v;B^{\frac{n}{2}}_{2,1}(\mathbb{R}^n_x)\big) \subsetneq L^\infty\Big(0,T;L^2_\alpha\big(\mathbb{R}^n_v;B^{\frac{n}{2}}_{2,1}(\mathbb{R}^n_x)\big)\Big),$$

besides, they are narrower than

$$L^2_\alpha\big(\mathbb{R}^n_v;B^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_x)\big),$$

and

$$L^\infty\Big(0,T;L^2_\alpha\big(\mathbb{R}^n_v;B^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_x)\big)\Big),$$

respectively, for $2 < p \le \infty$ in view of (1.6). Hence we show the uniqueness of a weak solution that belongs to the corresponding and slightly larger Bochner space for the solution of the Boltzmann equation obtained by the method of the integral equation. In particular, according to Ukai [31] and Ukai-Asano [33], the solution around the Maxwellian steady state is formulated by the corresponding integral equation. Then it is natural to consider the uniqueness of the solutions in the Bochner class rather than the smaller class such as in (1.7).

In what follows we use abbreviated notation of the norm of the Bochner spaces:

$$\|f\|_{L^2_{\alpha,v}(B^0_{p,1,x})} \equiv \|f\|_{L^2_\alpha\left(\mathbb{R}^n_v;B^0_{p,1}(\mathbb{R}^n_x)\right)},$$

$$\|f\|_{L^\infty_T\left(L^2_{\alpha,v}(B^0_{p,1,x})\right)} \equiv \|f\|_{L^\infty\left(0,T;L^2_\alpha\left(\mathbb{R}^n_v;B^0_{p,1}(\mathbb{R}^n_x)\right)\right)}.$$

§ 2. Scaling Invariance

If the collision term has the repulsive collision potential given by $U(|x|) = \frac{C}{|x|^{s-1}}$ for $s \in \mathbb{R}$, then by solving the Newton equation, the collision kernel is asymptotically given by

$$q(v - v_*, \omega) = C|v - v_*|^\gamma B(\cos\theta)$$

with

$$\gamma = \frac{s - 2(n-1) - 1}{s - 1} = 1 - \frac{2(n-1)}{s-1}, \quad s > 1,$$

where $\cos\theta = \frac{v - v_*}{|v - v_*|} \cdot \omega$. In general, the relation between the potential and the collision kernel in $n = 3$ is as follows (cf. Ukai [32]):

kinetic potential s	collision kernel γ	type	type of the equation
$s > 5$	$0 < \gamma < 1$	hard potential	Boltzmann
$s = 5$	$\gamma = 0$	Maxwell	Boltzmann
$2 < s < 5$	$-3 < \gamma < 0$	soft potential	Boltzmann
$s = 2$	$\gamma = -3$	Coulomb	Landau

If the collision kernel is given by the inverse power law as above, there exists an invariant scaling transform that maintain the equation invariant. For any $\lambda > 0$, let $\tilde{f}_\lambda(t, v, x) = \lambda^\sigma f(\lambda^a t, \lambda^b v, \lambda^c x)$ and if $\tilde{f}$ solves the equation (1.1), namely,

$$\partial_t \tilde{f} + v \cdot \nabla_x \tilde{f} = Q(\tilde{f}, \tilde{f})$$

then by setting $\tilde{v} = \lambda^b v$,

$$\begin{aligned} \lambda^{\sigma+a}\partial_t f + &\lambda^{\sigma+c-b}\tilde{v} \cdot \nabla_{\tilde{x}} f \\ &= \lambda^{2\sigma} \int_{\mathbb{R}^n} \int_{\mathbb{S}^{n-1}} C|v - v_*|^\gamma B(\cos\theta)\Big(f(v')f(v'_*) - f(v)f(v_*)\Big) d\omega dv_* \\ &= \lambda^{2\sigma - b\gamma - nb} \int_{\mathbb{R}^n} \int_{\mathbb{S}^{n-1}} C|\tilde{v} - \tilde{v}_*|^\gamma B(\cos\theta)\Big(f(\tilde{v}')f(\tilde{v}'_*) - f(\tilde{v})f(\tilde{v}_*)\Big) d\omega d\tilde{v}_*. \end{aligned}$$

Hence the necessary condition for the invariance is

$$a = c - b = \sigma - (\gamma + n)b$$

and then by setting $c = a + b$ and $\sigma = a + (\gamma + n)b$, we eliminate c and σ to have

$$\tilde{f}_\lambda(t, v, x) = \lambda^{a + (\gamma+n)b} f(\lambda^a t, \lambda^b v, \lambda^{a+b} x).$$

If $a = b = 1$, then

$$\tilde{f}_\lambda(t, v, x) = \lambda^{\gamma+n+1} f(\lambda t, \lambda v, \lambda^2 x).$$

In the invariant Bochner space

$$L^\theta\Big(\mathbb{R}_+; L^p_\alpha\big(\mathbb{R}^n_v; L^q(\mathbb{R}^n_x)\big)\Big)$$

with the homogeneous weight $|v|^\alpha$ in L^p_α, it follows

$$\frac{1}{\theta}+\frac{n+\alpha}{p}+\frac{2n}{q}=\gamma+n+1.$$

Since $0<\gamma<1$, the right hand side has to be $\gamma+n+1<5$ if $n=3$ and if $q=\theta=\infty$ and $p=2$, we see that

$$1+n<\frac{n+\alpha}{2}<2+n$$

and conclude that $2+n<\alpha<4+n$ for the case $0\le\gamma<1$.

§ 3. The Maxwellian steady state

Following Grad [13] and Ukai [29]-[31] (see also Ukai-Yang [34]), we consider the solution of the Boltzmann equation (1.1) around the Maxwell distribution and consider the initial disturbance from the steady state. We regard that the gas constant (Boltzmann constant) k and the absolute temperature T as $kT=1$, then the Maxwellian $\mu(v)=\mu_v\equiv\left(\frac{1}{2\pi}\right)^{\frac{n}{2}}e^{-\frac{|v|^2}{2}}$ satisfies

$$Q\big(\mu(v),\mu(v)\big)=0 \tag{3.1}$$

and hence it is a steady state of time dependent Boltzmann equation. Grad [13] and Ukai [29] considered the stability of the Maxwellian and showed the global existence of a solution to the perturbed problem near the Maxwellian. Besides, Ukai showed that the disturbance decays and the solution converges asymptotically to the Maxwellian. [1]

The solution of the Boltzmann equation has the three conservation laws: They are obtained by multiplying the equation by 1, v, $|v|^2$ and integrate it by parts, respectively with using the symmetry of the collision term $Q(f,f)$ (see for more [18]). On the other hand, the reversed assertion; $Q(f,f)=0$ implies $f=\mu(v)$ also holds essectially.

We consider the disturbance solution to the Boltzmann equation (1.1) around the Maxellian steady state. Setting

$$f\to\mu_v+\mu_v^{1/2}f$$

[1] To see the Maxwellian as the steady state, namely it satisfies (3.1) because

$$\mu_v(v'_*)\mu_v(v')=\frac{1}{2\pi}e^{-\frac{|v'_*|^2}{2}}\,\frac{1}{2\pi}e^{-\frac{|v'|^2}{2}}=\frac{1}{2\pi}e^{-\frac{|v_*|^2}{2}}\,\frac{1}{2\pi}e^{-\frac{|v|^2}{2}}=\mu_v(v_*)\mu_v(v).$$

we see that the Boltzmann equation can be reduced into the perturbation from the Maxwellian distribution:

$$\begin{cases} \partial_t f + v\cdot\nabla_x f - Lf = \sqrt{\mu_v}^{-1} Q(\sqrt{\mu_v} f, \sqrt{\mu_v} f), & t>0, (v,x)\in\mathbb{R}^n\times\mathbb{R}^n, \\ f(0,v,x) = f_0(v,x), & (v,x)\in\mathbb{R}^n\times\mathbb{R}^n, \end{cases} \tag{3.2}$$

where L denotes the nonlinear linear dissipation operator given by

$$\begin{aligned} Lf =& \sqrt{\mu_v}^{-1}\Big(Q(\mu_v, \sqrt{\mu_v} f) + Q(\sqrt{\mu_v} f, \mu_v)\Big) \\ =& \sqrt{\mu_v}^{-1} Q(\mu_v, \sqrt{\mu_v} f) + \sqrt{\mu_v}^{-1} Q(\sqrt{\mu_v} f, \mu_v) \\ \equiv& \Gamma(\mu_v, f) + \Gamma(f, \mu_v) \end{aligned} \tag{3.3}$$

with

(3.4)

$$\Gamma(\mu_v, f) = \int_{\mathbb{R}^n}\int_{\mathbb{S}^{n-1}} q(|v-v_*|,\omega)\big(\sqrt{\mu(v)}^{-1}\mu(v_*')\sqrt{\mu(v')}f(v') - \mu(v_*)f(v)\big)d\omega dv_*,$$

$$\Gamma(f, \mu_v) = \int_{\mathbb{R}^n}\int_{\mathbb{S}^{n-1}} q(|v-v_*|,\omega)\big(\sqrt{\mu(v)}^{-1}\sqrt{\mu(v_*')}f(v_*')\mu(v') - \sqrt{\mu(v_*)\mu(v)}f(v_*)\big)d\omega dv_*.$$

The kernel of the operator L is expressed by the linear hall of the bases:

$$ker(-L) = L.H.\{\sqrt{\mu(v)}, v\sqrt{\mu(v)}, |v|^2\sqrt{\mu(v)}\}$$

then we introduce the projection operator P by

$$Pf \equiv \Big(a(t,x) + v\cdot b(t,x) + (|v|^2-3)c(t,x)\Big)\sqrt{\mu(v)},$$

where $a(t,x) = (\sqrt{\mu_v}, f)$, $b(t,x) = (v\sqrt{\mu_v}, f)$, $c(t,x) = (\sqrt{\mu_v}, f)$ are projection components.

It is then known that the linear operator is subject to the following dissipative estimate:

$$\int_{\mathbb{R}^n} f(t,x,v)(-L)f(t,x,v)dv \ge \lambda_0 \int_{\mathbb{R}^n} \nu(v)|(1-P)f|^2 dv, \tag{3.5}$$

where the weight function $\nu(v)$ is given by

$$\nu(v) = \int_{\mathbb{R}^n}\int_{\mathbb{S}^{n-1}} q(|v-v_*|,\omega)\sqrt{\mu_{v_*}}d\omega dv_* \simeq (1+|v|)^\gamma.$$

Hence this term produces the energy dissipation of the solution.

On the other hand the nonlinear term of (3.2) is now given by the following: Noting

$|v|^2+|v_*|^2=|v'|^2+|v_*'|^2$ and $\mu(v)\mu(v_*)=\mu(v')\mu(v_*')$,

$$
\begin{aligned}
\Gamma(f,g)\equiv&\sqrt{\mu_v}^{-1}Q(\sqrt{\mu_v}f,\sqrt{\mu_v}g)\\
=&\sqrt{\mu(v)}^{-1}\int_{\mathbb{R}^n}\int_{\mathbb{S}^{n-1}}q(|v-v_*|,\omega)\Big\{\sqrt{\mu(v_*')}\sqrt{\mu(v')}f(v_*')g(v')\\
&\qquad\qquad-\sqrt{\mu(v_*)}\sqrt{\mu(v)}f(v_*)g(v)\Big\}d\omega dv_*\\
=&\int_{\mathbb{R}^n}\int_{\mathbb{S}^{n-1}}q(|v-v_*|,\omega)\sqrt{\mu(v_*)}\Big\{f(v_*')g(v')-f(v_*)g(v)\Big\}d\omega dv_*.
\end{aligned}
\tag{3.6}
$$

By setting the linear operator by $\Lambda\equiv v\cdot\nabla_x-L$, and using the semigroup $\{e^{-t\Lambda}\}_{t\geq0}$ generated by the operator Λ, the solution to the Boltzmann equation (3.2) around the perturbation of the Maxwellian is now given by the solution of the corresponding integral equation:

$$
\begin{aligned}
f(t)=&e^{-t\Lambda}f_0+\int_0^t e^{-(t-s)\Lambda}\sqrt{\mu_v}^{-1}Q(\sqrt{\mu_v}f(s),\sqrt{\mu_v}f(s))ds\\
=&e^{-t\Lambda}f_0+\int_0^t e^{-(t-s)\Lambda}\Gamma(f(s),f(s))ds.
\end{aligned}
\tag{3.7}
$$

Definition (A (weak) mild solution). Let $n=2,3$, $1\leq p\leq\infty$, $1\leq q<\infty$ and $\alpha\geq0$. For any initial data $f_0\in L^2_\alpha\big(\mathbb{R}^n_v;B^0_{p,1}\cap B^{\frac{n}{q}}_{q,1}(\mathbb{R}^n_x)\big)$, f is a (weak) mild solution to the Boltzmann equation (3.2), if for some $T>0$, $f\in C\big([0,T);L^2_\alpha(\mathbb{R}^n_v;B^0_{p,1}\cap B^{\frac{n}{q}}_{q,1}(\mathbb{R}^n_x)\big)$ and it satisfies the integral equation (3.7) in $C\big([0,T);L^2_\alpha(\mathbb{R}^n_x;B^0_{p,1}\cap B^{\frac{n}{q}}_{q,1}(\mathbb{R}^n_x)\big)$.

The analogous definition of weak mild solutions is also considered for the homogeneous Besov spaces $\dot B^0_{p,1}(\mathbb{R}^n_x)\cap\dot B^{\frac{n}{q}}_{q,1}(\mathbb{R}^n_x)$.

If the exponent $q\leq n$, then the above defined mild solution is a strong solution. Duan et al. [12] showed the existence and uniqueness of a strong solution of the Boltzmann equation in the Chemin-Lerner space based on the inhomogeneous Besov space; $\widetilde{L^\infty(0,T};L^2_\alpha(\mathbb{R}^n_v;B^{\frac{n}{2}}_{2,1}(\mathbb{R}^n))$ for $n=3$ and they employed the energy method and the Littlewood-Paley decomposition with the para-product formula. It is interesting if we can regard such solution as a solution of the integral equation (3.7). In fact, to justify the integral equation in the sense of Bochner integral, it is natural to introduce the Bochner spaces into the existence and uniqueness of the solution unless we regard the time integration upon the space-velocity space such as $L^2(\mathbb{R}^n_v\times\mathbb{R}^n_x)$ or $L^\infty(\mathbb{R}^n_v\times\mathbb{R}^n_x)$. We concentrate our attention on the function space for spatial variable $x\in\mathbb{R}^n$ and introduce the critical homogeneous Besov spaces to treat the integral equation (3.7).

§ 4. Bilinear estimate in the critical Besov space

To treat the nonlinear collision term, proper bilinear estimates are required. We first note that the Hölder type inequalities such as

$$\|fg\|_{\dot{B}^0_{p,1}} \le C\|f\|_\infty \|g\|_{\dot{B}^0_{p,1}}$$

in the Besov space generally fails. To show an appropriate bilinear estimate, we adopt the following Hölder-type inequality for a function in $B^{n/q}_{q,1}(\mathbb{R}^n)$, which plays the role for the substitute for the L^∞ space. The analogous estimate for the homogeneous Besov spaces has been established ([25]). Let $1 \le p \le \infty$ and $1 \le q < \infty$. Then for any $f \in \dot{B}^{\frac{n}{q}}_{q,\infty} \cap \dot{B}^0_{\infty,1}$ and $g \in \dot{B}^0_{p,1}$, there exists $C > 0$ such that

$$\|fg\|_{\dot{B}^0_{p,1}} \le C\|f\|_{\dot{B}^{\frac{n}{q}}_{q,1}} \|g\|_{\dot{B}^0_{p,1}}. \tag{4.1}$$

We show the inhomogeneous version of the bilinear estimate (4.1).

Proposition 4.1. *Let $1 \le p \le \infty$ and $1 \le q < \infty$. For $f \in B^{\frac{n}{q}}_{q,\infty}(\mathbb{R}^n) \cap B^0_{\infty,1}(\mathbb{R}^n)$ and $g \in B^0_{p,1}(\mathbb{R}^n)$ there exists $C > 0$ such that*

$$\|fg\|_{B^0_{p,1}} \le C\big(\|f\|_{B^{\frac{n}{q}}_{q,\infty}} + \|f\|_{B^0_{\infty,1}}\big)\|g\|_{B^0_{p,1}}. \tag{4.2}$$

In particular, there exists $C > 0$ such that

$$\|fg\|_{B^0_{p,1}} \le C\|f\|_{B^{\frac{n}{q}}_{q,1}} \|g\|_{B^0_{p,1}}. \tag{4.3}$$

Proof of Proposition 4.1. The estimate (4.3) is a direct consequence of (4.2) by the embedding $B^{\frac{n}{q}}_{q,1} \subset B^{\frac{n}{q}}_{q,\infty} \cap B^0_{\infty,1}$ for any $1 \le q < \infty$. To show (4.2), we start with Bony's para-product formula: For $f \in B^0_{\infty,1}$ and $g \in B^0_{p,1}$, it follows that

$$\begin{aligned} f(x)\ g(x) =& \sum_{k\in\bar{\mathbb{N}}} (\phi_k * f(x))(P_k g(x)) + \sum_{k\in\bar{\mathbb{N}}} (P_k f(x))(\phi_k * g(x)) \\ &+ \sum_{k\in\bar{\mathbb{N}}} \sum_{|\ell-k|\le 2} (\phi_k * f(x))(\phi_\ell * g(x)) \\ \equiv& h_1 + h_2 + h_3. \end{aligned} \tag{4.4}$$

Here $P_k g = \sum_{-1\le\ell\le k-3} \phi_\ell * g = \psi_{2^{-(k-3)}} * g \quad (\lambda > 0)$, where $\psi_\lambda = \lambda^{-n}\psi(x/\lambda)$ for $\lambda > 0$, and

$$\operatorname{supp} \mathcal{F}((\phi_k * f)(P_k g)) \subset \{\xi \in \mathbb{R}^n; 2^{k-2} \le |\xi| \le 2^{k+2}\}.$$

The Young and Minkowski inequalities imply that for $1/p_1 + 1/p_2 = 1/p$

$$
\begin{aligned}
\|h_1\|_{B^0_{p,1}} =& \sum_{j\in\bar{\mathbb{N}}} \Big\| \sum_{|k-j|\le 2} \phi_j * ((\phi_k * f)(P_k g)) \Big\|_p \\
\le& \sum_{j\in\bar{\mathbb{N}}} \sum_{|\ell|\le 2} \Big\| \phi_j * ((\phi_{j+\ell} * f)(P_{j+\ell} g)) \Big\|_p \\
\le& \sum_{j\in\bar{\mathbb{N}}} \sum_{|\ell|\le 2} \|\phi_j\|_1 \|\phi_{j+\ell} * f\|_{p_1} \|P_{j+\ell} g\|_{p_2} \qquad (4.5)\\
\le& C\|g\|_{p_2} \sum_{j\in\bar{\mathbb{N}}} \sum_{|\ell|\le 2} \|\phi_{j+\ell} * f\|_{p_1} \\
\le& C\|g\|_{p_2} \|f\|_{B^0_{p_1,1}},
\end{aligned}
$$

where we understand $\phi_{j+\ell} = \psi$ if $j+\ell < 0$. Exchanging f and g we obtain the estimate for the second term of (4.4) and for $1/q_1 + 1/q_2 = 1/p$

$$
\|h_2\|_{B^0_{p,1}} \le C\|f\|_{q_1} \|g\|_{B^0_{q_2,1}}. \tag{4.6}
$$

Here we may choose (p_1, p_2) and (q_1, q_2) independently. To obtain the third term in (4.4), we notice that

$$
\operatorname{supp} \mathcal{F}(\phi_k * f \cdot \phi_\ell * g) \subset \{\xi \in \mathbb{R}^n; |\xi| \le 2^{\max\{k,\ell\}+2}\}
$$

and

$$
\phi_j * ((\phi_k * f)(\phi_\ell * g)) = 0, \quad \text{where} \quad \max\{k,\ell\} \le j-3.
$$

Let $\tilde{\phi}_k = \sum_{|k-j|\le 2} \phi_j$. By changing $k = j + k'$, we see $k \ge j-2$ with $k' \ge -2$ and it follows that

$$
\begin{aligned}
\|h_3\|_{B^0_{p,1}} =& \sum_{j\in\bar{\mathbb{N}}} \Big\| \sum_{j-2\le\max\{k,\ell\}} \sum_{|k-\ell|\le 2} \phi_j * ((\phi_k * f)(\phi_\ell * g)) \Big\|_p \\
\le& \sum_{j\in\bar{\mathbb{N}}} \sum_{k'\ge -2} \|\phi_j\|_r \|\phi_{j+k'} * f\|_{q_1} \|\widetilde{\phi}_{j+k'} * g\|_{q_2},
\end{aligned}
$$

where q_1 and q_2 satisfy

$$
\frac{1}{p} + 1 = \frac{1}{r} + \frac{1}{q_1} + \frac{1}{q_2}. \tag{4.7}
$$

Noting

$$
\|\phi_j\|_r = 2^{nj(1-\frac{1}{r})}\|\phi\|_r,
$$

$$
\begin{aligned}
\|h_3\|_{B^0_{p,1}} \le & C\sum_{j\in\bar{\mathbb{N}}}\sum_{k'\ge -2} 2^{\frac{nj}{r'}}\|\phi_{j+k'}*f\|_{q_1}\|\widetilde{\phi}_{j+k'}*g\|_{q_2} \\
\le & C\sum_{j\in\bar{\mathbb{N}}}\sum_{k'\ge -2} 2^{\frac{nj}{r'}}2^{-\frac{nk'}{r'}}2^{\frac{nk'}{r'}}\|\phi_{j+k'}*f\|_{q_1}\|\widetilde{\phi}_{j+k'}*g\|_{q_2} \\
\le & C\sup_{j,k} 2^{\frac{n}{r'}(j+k')}\|\phi_{j+k'}*f\|_{q_1}\sum_{j\in\bar{\mathbb{N}}}\Big(\sum_{k'\ge -2} 2^{-\frac{n}{r'}k'}\Big)\|\widetilde{\phi}_{j+k'}*g\|_{q_2} \\
\le & C\|f\|_{B^{\frac{n}{r'}}_{q_1,\infty}}\|g\|_{B^0_{q_2,1}}.
\end{aligned}
$$

Note that we require $r' < \infty$ for the convergence of the last term. From (4.7),

$$
\frac{1}{r'} = 1 - \frac{1}{r} = \frac{1}{q_1} + \frac{1}{q_2} - \frac{1}{p}
$$

and choosing $q_2 = p$, $q_1 = q$, we see from $\dfrac{1}{r'} = \dfrac{1}{q}$ that

$$
\|h_3\|_{B^0_{p,1}} \le C\|f\|_{B^{\frac{n}{q}}_{q,\infty}}\|g\|_{B^0_{p,1}}, \tag{4.8}
$$

where we require $q < \infty$. Letting $p_1 = \infty$, $p_2 = p$ in (4.5) and letting $q_1 = \infty$, $q_2 = p$ in (4.6) and combining with (4.8), it follows from (4.4) that

$$
\begin{aligned}
\|fg\|_{B^0_{p,1}} \le & \|h_1\|_{B^0_{p,1}} + \|h_2\|_{B^0_{p,1}} + \|h_3\|_{B^0_{p,1}} \\
\le & C\|f\|_{B^0_{\infty,1}}\|g\|_p + C\|f\|_\infty\|g\|_{B^0_{p,1}} + C\|f\|_{B^{\frac{n}{q}}_{q,\infty}}\|g\|_{B^0_{p,1}} \\
\le & C(\|f\|_{B^0_{\infty,1}} + \|f\|_{B^{\frac{n}{q}}_{q,\infty}})\|g\|_{B^0_{p,1}}.
\end{aligned}
$$

□

The space $B^{n/q}_{q,1}(\mathbb{R}^n)$ and $\dot{B}^{n/q}_{q,1}(\mathbb{R}^n)$ has a nice embedding property. Let

$$
C_v(\mathbb{R}^n) = \big\{f \in C(\mathbb{R}^n);\ |f(x)| \to 0 \text{ as } |x| \to \infty\big\}.
$$

Proposition 4.2. *Let $1 \le q < \infty$ and $\mathcal{S}(\mathbb{R}^n)$ be the rapidly decreasing smooth functions. Then*

$$
\mathcal{S}(\mathbb{R}^n) \hookrightarrow \dot{B}^{n/q}_{q,1}(\mathbb{R}^n) \hookrightarrow B^{n/q}_{q,1}(\mathbb{R}^n) \hookrightarrow C_v(\mathbb{R}^n).
$$

In particular, the embedding of the left-hand side is dense.

For the proof, see [25].

Remark 1. It is known that $B^0_{\infty,1}(\mathbb{R}^n) \subset BUC(\mathbb{R}^n)$ but $B^0_{\infty,1}(\mathbb{R}^n) \not\subset C_v(\mathbb{R}^n)$, where $BUC(\mathbb{R}^n)$ is a set of uniformly continuous and bounded functions. For instance, $\sin x \in B^0_{\infty,1}(\mathbb{R})$ and it is in $BUC(\mathbb{R})$ but not in $C_v(\mathbb{R})$.

§ 5. Maximal regularity for a weight operator

The spectral analysis for the linearized Boltzmann equation is considered in detail in Ukai-Yang [34]. Let

$$\Lambda \equiv v \cdot \nabla_x - L,$$

where the operator L is appeared from the collision term and is given by (3.3) and (3.4). The structure of the operator Λ and hence the evolution semi-group $\{e^{-t\Lambda}\}_{t\geq 0}$ is analyzed in [34]. In particular, they showed that the operator $-L$ can be decomposed into the weight operator $\nu(v)$ given by (1.5) and a compact self-adjoint operator $-K$ on $L^2(\mathbb{R}^n_v; L^2(\mathbb{R}^n_x))$ (cf. [34, Proposition 2.1, Lemma 2.3]) [2] . According to [31] and [34], it is known that there exists positive number σ_0 such that the spectrum of the operator $-\widehat{\Lambda(v,\xi)}$ in the right half complex plane $Rez \geq -\sigma_0$ consists only of $n+2$ discrete semi-simple eigenvalues $\lambda_j(\xi)$ with negative real parts for $j = 0, 1, 2, \cdots, n+1$. Hence letting $P_j(\xi)$ as the corresponding eigenprojections, the Fourier image of the evolution operator in x variable can be given for some $\kappa_0 > 0$ that

$$e^{-t\widehat{\Lambda}} \equiv e^{-(-iv\cdot\xi+\nu(v))t} + \sum_{j=0}^{n+2} \Phi_j(t,v,\xi)\mathcal{F}_{x\to\xi} \tag{5.1}$$

with

$$\begin{aligned}
\Phi_j(t,v,\xi) &= e^{\lambda_j(\xi)t}P_j(\xi)\chi_{|\xi|<\kappa_0}, \qquad j = 0, 1, \cdots, n+1, \\
\Phi_{n+2}(t,v,\xi) &= \int_{-\sigma_0-i\infty}^{-\sigma_0+i\infty} e^{\lambda t}(\lambda - A)^{-1}(I - K(\lambda - A)^{-1})^{-1}K(\lambda - A)^{-1}d\lambda,
\end{aligned}$$

where we set $A = -v \cdot \nabla_x - \nu(v)$. By using the notation above, $E_1(t)$ and $E_2(t)$ are given by

$$\begin{cases} E_1(t) \equiv \displaystyle\sum_{j=0}^{n+1} \mathcal{F}^{-1}_{\xi\to x}\Phi_j(t,v,\xi)\mathcal{F}_{x\to\xi}, \\ E_2(t) \equiv \mathcal{F}^{-1}_{\xi\to x}\Phi_{n+2}(t,v,x)\mathcal{F}_{x\to\xi}. \end{cases} \tag{5.2}$$

Thus by (5.1), the evolution operator $e^{-t\Lambda}$ can be decomposed into three parts:

$$e^{-t\Lambda}f \equiv e^{-(v\cdot\nabla_x+\nu(v))t}f + E_1(t)f + E_2(t)f, \tag{5.3}$$

where $E_1(t)f$ and $E_2(t)f$ are derived from the spectrum of the operator given in (5.2).

In order to recover the velocity weight loss, we invoke maximal regularity for the evolution operator. Let $A_\varsigma f = \mathcal{F}^{-1}_{v\to\varsigma}[\nu(v)\widehat{f}(v)]$, where ν is given by (1.5). $\widehat{f}$ and $\mathcal{F}^{-1}_{v\to\varsigma}[\,\cdot\,]$

[2]Here we remark that $-\Lambda = B$ in the notation of [34].

denotes the Fourier and the Fourier inverse transform between v and ς, respectively. We then introduce maximal regularity for the evolution operator $\{e^{-tA_\varsigma}\}_{t\in I}$ with $I=(0,T)$ for some $T<\infty$. If we consider the operator in the Hilbert space $L^2_\varsigma(\mathbb{R}^n)$ then the following estimate is known (cf. Prüss-Simonett [26, Proposition 4.1.8]).

Proposition 5.1 ([26]). *Let $1<r<\infty$, $\alpha\ge 0$ and the operator A_ς is defined as above in $L^2_\varsigma(\mathbb{R}^n)$. Then A_ς has maximal regularity in $L^p(I;H^\alpha(\mathbb{R}^n))$. Namely for any $\alpha\ge 0$,*

$$\left\|\int_0^t A_\varsigma e^{-(t-s)A_\varsigma}h(s)ds\right\|_{L^r(I;H^\alpha(\mathbb{R}^n_\varsigma))}\le C\|h\|_{L^r(I;H^\alpha(\mathbb{R}^n_\varsigma))}.$$

As a bi-product of Proposition 5.1, we obtain a useful weighted estimate for the evolution $e^{-t\nu(v)}$ via the Plancherel identity.

Proposition 5.2. *Let $1<r<\infty$, $\alpha\ge 0$, $0\le\gamma<1$ and $\nu(v)$ is defined by* (1.5)*. Then the evolution operator of ν has the following estimate:*

$$\left\|\int_0^t e^{-(t-s)\nu(v)}\nu(v)g(s)ds\right\|_{L^r(I;L^2_\alpha(\mathbb{R}^n_v))}\le C\|g\|_{L^r(I;L^2_\alpha(\mathbb{R}^n_v))}.$$

We then apply Proposition 5.2 to the operator A defined in the above. Let X be a Banach space for x variable with the shifting invariance. Namely, for any $y\in\mathbb{R}^n$,

$$\|h(\cdot-y)\|_X=\|h(\cdot)\|_X.$$

Then since by Ukai-Yang [34]

$$e^{-tA}h(v,x)=e^{-t\nu(v)}h(v,x-vt) \tag{5.4}$$

and the multiplication of the weight $\nu(v)$ is non-negative, one can find that

$$\begin{aligned}\|e^{-tA}h(v,x)\|_{L^2_v(\mathbb{R}^n;X)}=&\|e^{-t\nu(v)}h(v,x-vt)\|_{L^2_v(\mathbb{R}^n;X)}\\ \le&\left\|e^{-t\nu(v)}\|h(v,\cdot-vt)\|_X\right\|_{L^2_v(\mathbb{R}^n)}\\ =&\left\|e^{-t\nu(v)}\|h(v,\cdot)\|_X\right\|_{L^2_v(\mathbb{R}^n)}.\end{aligned}$$

Proposition 5.3. *Let X be the shifting invariant Banach space. Let $1<r<\infty$ and $\alpha\ge 0$. Then there exists a positive constant $C_r=C(r,n)>0$ such that*

$$\left\|\int_0^t \nu(v)e^{-(t-s)A}g(s)ds\right\|_{L^r(I;L^2_{\alpha,v}(X))}\le C_r\|g\|_{L^r(I;L^2_{\alpha,v}(X))}$$

for all $g\in L^r(I;L^2_{\alpha,v}(X))$.

Proof of Proposition 5.3. For $g \in L^r(I; L^2_{\alpha,v}(X))$, let $H(s,v) \equiv \|g(s,v,\cdot)\|_X$. Then applyting Proposition 5.2, it directly follows that

$$\begin{aligned}
&\left\| \int_0^t \nu(v) e^{-(t-s)A} g(s) ds \right\|_{L^r(I; L^2_{\alpha,v}(X))} \\
&= \left\| \int_0^t \nu(v) e^{-(t-s)\nu(v)} g(s,v,\cdot - v(t-s)) ds \right\|_{L^r(I; L^2_{\alpha,v}(X))} \\
&\leq \left\| \int_0^t \nu(v) e^{-(t-s)\nu(v)} \|g(s,v,\cdot)\|_X ds \right\|_{L^r(I; L^2_{\alpha,v})} \\
&= \left\| \int_0^t \nu(v) e^{-(t-s)\nu(v)} H(s,v) ds \right\|_{L^r(I; L^2_{\alpha,v})} \\
&\leq C_r \|H\|_{L^r(I; L^2_{\alpha,v})} = C_r \|g\|_{L^r(I; L^2_{\alpha,v}(X))}.
\end{aligned}$$

□

Remark 2. Let X be a Banach space in x variable and suppose that $L^2_{\alpha,\varsigma}(X_x)$ is UMD (unconditional martingale differences). Then the general theory of maximal regularity implies the desired estimate such as

$$\begin{aligned}
&\left\| \mathcal{F}_{v\to\varsigma} \Big[\int_0^t \nu(v) e^{-(t-s)A} g(t,v,x) ds \Big] \right\|_{L^r(I; L^2_{\alpha,\varsigma}(X))} \\
&= \left\| \int_0^t \nu(D_\varsigma) e^{-(t-s)\nu(D_\varsigma)} \widetilde{g}(s,\varsigma,x) ds \right\|_{L^r(I; H^\alpha_\varsigma(X))} \\
&\leq C_r \|\widetilde{g}(s,\varsigma,x)\|_{L^r(I; H^\alpha_\varsigma(X))} = C_r \|g(s,\varsigma,x)\|_{L^r(I; L^2_{\alpha,\varsigma}(X))}
\end{aligned}$$

for all $1 < r < \infty$ and $\alpha \geq 0$, where $\widetilde{g}(s,\varsigma,x) = \mathcal{F}_{v\to\varsigma} g(s,v,x)$. When $X = B^0_{p,1}$ or $B^s_{p,1}$, however, $L^2_{\alpha,\varsigma}(X_x)$ is not reflexive and hence it is not UMD (cf. Amann [5]) and the standard theory is not applicable. Proposition 5.3 overcome this difficulty for our setting by using the special structure (5.4) of the evolution operator e^{-tA}.

Since the Banach spaces $B^0_{p,1}$ and $B^{\frac{n}{p}}_{p,1}$ are shifting invariant, by virtue of Proposition 5.2, we conclude that the evolution operator $e^{-t\Lambda}$ has a suitable estimate with the velocity weight (cf. Ukai-Yang [34] Lemma 2.10, Theorem 2.18).

Proposition 5.4. *Let $1 < r < \infty$, $1 \leq p \leq \infty$, $1 \leq q < \infty$, $\alpha \geq 0$, $0 \leq \gamma < 1$ and $e^{-t\Lambda}$ be the linear evolution operator of the linearized Boltzmann equations. Let X be either $B^{\frac{n}{q}}_{q,1}(\mathbb{R}^n_x)$ or $B^0_{p,1}(\mathbb{R}^n_x)$. Then there exists a constant $C > 0$ such that for any $G \in L^r\big(0,T; L^2_\alpha(\mathbb{R}^n_v; X)\big)$,*

$$\left\| \int_0^t e^{-(t-s)\Lambda} \nu(v) G(s) ds \right\|_{L^r\left(I; L^2_\alpha(\mathbb{R}^n_v; X)\right)} \leq C \|G\|_{L^r\left(I; L^2_\alpha(\mathbb{R}^n_v; X)\right)}$$

for any $\alpha \geq 0$ *and* $I = (0, T)$*, where the weight function* $\nu(v)$ *is given by* (1.5)*.*

The proof of Proposition 5.4 is essentially given in [34] by using the evolution expression (5.3) and (5.2) except the weight gain estimate for the operator $A = -v \cdot \nabla_x - \nu(v)$. Note that the compact perturbation part can be treated in the Besov space since they are L^p and hence $B^0_{p,1}$ bounded operators. The advantage of the result due to Duan et al. [12] is that by employing the energy type estimate in the Chemin-Lerner spaces, the gain of the weight $\nu(v)$ is maximal (cf. (3.5)). The above estimate shows such a gain is also holding for the formulation of the integral equations in a Bochner space.

§ 6. Uniqueness for the weak mild solution in a critical space

The nonlinear interaction for the Boltzmann equation (3.7) is given by (3.6) and we set

$$
\begin{aligned}
\Gamma(f,g) =& \sqrt{\mu_v}^{-1} Q(\sqrt{\mu_v} f, \sqrt{\mu_v} g) \\
=& \int_{\mathbb{R}^n} \int_{\mathbb{S}^{n-1}} q(|v - v_*|, \omega) \sqrt{\mu(v_*)} \Big\{ f(v'_*) g(v') - f(v_*) g(v) \Big\} d\omega dv_* \\
\equiv& \Gamma_1(f,g) - \Gamma_2(f,g).
\end{aligned}
\tag{6.1}
$$

We then show the uniqueness of the weak solution for the integral equation (3.7) in the class $f \in C\big(I; L^2_\alpha(\mathbb{R}^n_v; B^0_{p,1} \cap B^{\frac{n}{q}}_{q,1}(\mathbb{R}^n_x))\big)$ with $1 \leq p \leq \infty$, $1 \leq q < \infty$ following the method of Ukai-Yang [34].

Theorem 6.1. *Let* $\alpha \geq 0$*,* $1 \leq p \leq \infty$ *and* $1 \leq q < \infty$*. Assume that the collision kernel satisfies the estimate* (1.4) *with* $0 < \gamma < 1$*. Then if a (weak) mild solution of the Cauchy problem to the Boltzmann equation around the Maxellian in*

$$
f \in C\Big([0,T]; L^2_\alpha\big(\mathbb{R}^n_v; B^0_{p,1} \cap B^0_{\infty,1} \cap B^{\frac{n}{q}}_{q,\infty}(\mathbb{R}^n_x)\big)\Big)
$$

exists for some $T > 0$ *and there exists a small constant* $M_0 > 0$ *such that*

$$
\|f\|_{L^\infty\big(0,T; L^2_{\alpha,v}(B^0_{p,1,x} \cap B^0_{\infty,1,x} \cap B^{\frac{n}{q}}_{q,\infty,x})\big)} \leq M_0,
$$

then it is at most one.

In view of the bi-linear estimate, the following uniqueness criterion is one of a corollary from Theorem 6.1.

Corollary 6.2. *Let $\alpha \geq 0$, $1 \leq p \leq \infty$ and $1 \leq q < \infty$. Then if a weak mild solution of the Cauchy problem to the Boltzmann equation around the Maxellian in*

$$f \in C\Big([0,T]; L^2_\alpha\big(\mathbb{R}^n_v; B^0_{p,1} \cap B^{\frac{n}{q}}_{q,1}(\mathbb{R}^n_x)\big)\Big)$$

exists for any $T > 0$ and there exists a small constant $M_0 > 0$ such that

$$\|f\|_{L^\infty\big(0,T; L^2_{\alpha,v}(B^0_{p,1,x} \cap B^{\frac{n}{q}}_{q,1,x})\big)} \leq M_0,$$

then it is at most one.

Remark 3. The class introduced by Ukai-Yang [34] is $C\big([0,T]; L^2(\mathbb{R}^n_v \times \mathbb{R}^n_x) \cap L^\infty_\beta(\mathbb{R}^n_v \times \mathbb{R}^n_x)\big)$, where $L^\infty_\beta(\mathbb{R}^n_v \times \mathbb{R}^n_x)$ is weighted L^∞ space with v-weight $\langle v\rangle^\beta$. There is no inclusion relation between their class and the above class.

Remark 4. The solution class in Corollary 6.2 includes the class of the result due to Duan et al. [12].

The following estimate is for the linear evolution operator with the weight function $\nu(v)$ which originally due to Ukai-Yang [34] in Lemma 2.10 and Theorem 2.18. We modify the estimate in the spatial function spaces as the inhomogeneous Besov spaces without an essential change.

Proof of Theorem 6.1. We show the proof of Corollary 6.2 for simplicity. The proof for Theorem 6.1 can be obtained by arranging the corresponding Besov norms with a usage of Proposition 4.1. We start from the estimate for the bi-linear term $\Gamma_1(f,g)$ defined in (6.1): For any $t > 0$, $\alpha \geq 0$ and $0 \leq \delta \leq 1$,

(6.2)

$$\begin{aligned}
&\|\nu(v)^{-\delta}\Gamma_1(f,g)\|_{L^2_\alpha(\mathbb{R}^n_v; B^0_{p,1}(\mathbb{R}^n_x))} \\
&\quad = \Big\|\nu(v)^{-\delta} \int_{\mathbb{R}^n}\int_{\mathbb{S}^{n-1}} q(|v-v_*|,\omega)\sqrt{\mu(v_*)} f(v'_*)g(v') d\omega dv_*\Big\|_{L^2_{\alpha,v}(B^0_{p,1,x})} \\
&\quad \leq \Big\|\nu(v)^{-\delta} \int_{\mathbb{R}^n}\int_{\mathbb{S}^{n-1}} q(|v-v_*|,\omega)\sqrt{\mu(v_*)} \big\|f(v'_*)g(v')\big\|_{B^0_{p,1,x}} d\omega dv_*\Big\|_{L^2_{\alpha,v}} \\
&\quad \leq C\Big\|\nu(v)^{-\delta} \int_{\mathbb{R}^n}\int_{\mathbb{S}^{n-1}} q(|v-v_*|,\omega)\sqrt{\mu(v_*)} \big\|f(v'_*)\big\|_{B^0_{p,1,x}} \big\|g(v')\big\|_{B^{\frac{n}{q}}_{q,1,x}} d\omega dv_*\Big\|_{L^2_{\alpha,v}} \\
&\quad \equiv I_1.
\end{aligned}$$

The last term I_1 in (6.2) can be estimated as follows:

(6.3)
$$\begin{aligned}
I_1 \equiv &\bigg(\int_{\mathbb{R}^n_v} \langle v\rangle^{2\alpha}\nu(v)^{-2\delta} \\
&\qquad \times \Big| \int_{\mathbb{R}^n}\int_{\mathbb{S}^{n-1}} q(|v-v_*|,\omega)\sqrt{\mu(v_*)}\big\|f(v_*')\big\|_{B^0_{p,1,x}}\big\|g(v')\big\|_{B^{\frac{n}{q}}_{q,1,x}} d\omega dv_*\Big|^2 dv\bigg)^{1/2} \\
\le & C\bigg[\int_{\mathbb{R}^n_v}\langle v\rangle^{2\alpha-2\delta\gamma}\bigg\{\int_{\mathbb{R}^n_{v_*}}\int_{\mathbb{S}^{n-1}}\Big(q(|v-v_*|,\omega)\sqrt{\mu(v_*)}\Big)^2 d\omega dv_* \\
&\qquad \times \int_{\mathbb{R}^n_{v_*}}\int_{\mathbb{S}^{n-1}}\big\|f(v_*')\big\|^2_{B^0_{p,1,x}}\big\|g(v')\big\|^2_{B^{\frac{n}{q}}_{q,1,x}} d\omega dv_*\bigg\}^{\frac12\cdot 2} dv\bigg]^{1/2} \\
\le & C|\omega_n|\max_\theta|B_0(\theta)|\bigg[\int_{\mathbb{R}^n_v}\langle v\rangle^{2\alpha-2\delta\gamma}\Big(\int_{\mathbb{R}^n_{v_*}}|v-v_*|^{2\gamma}\mu(v_*)dv_*\Big) \\
&\qquad \times \int_{\mathbb{R}^n_{v_*}}\int_{\mathbb{S}^{n-1}}\big\|f(v_*')\big\|^2_{B^0_{p,1,x}}\big\|g(v')\big\|^2_{B^{\frac{n}{q}}_{q,1,x}} d\omega dv_* dv\bigg]^{1/2} \equiv I_1^1.
\end{aligned}$$

Here we invoke the well-known estimate for the velocity weight;

$$\int_{\mathbb{R}^n_{v_*}} |v-v_*|^{2\gamma}\mu(v_*)dv_* \le C\langle v\rangle^{2\gamma} \tag{6.4}$$

by (1.3), $|v-v_*| = |v'-v_*'|$ and hence

$$|v| = |v' + (v-v_*,\omega)\omega| \le |v'| + |v'-v_*'|.$$

Therefore since $\langle v\rangle \le C(\langle v'\rangle + \langle v_*'\rangle)$ a change of variable $(v,v_*)\to(v',v_*')$ implies

(6.5)
$$\begin{aligned}
I_1^1 \le & C|\omega_n|\Big(\int_{\mathbb{R}^n_v}\Big(\int_{\mathbb{R}^n_{v_*}}\langle v\rangle^{2\alpha+2(1-\delta)\gamma}\big\|f(v_*')\big\|^2_{B^0_{p,1,x}}\big\|g(v')\big\|^2_{B^{\frac{n}{q}}_{q,1,x}} dv_*\Big)dv\Big)^{1/2} \\
\le & C|\omega_n|\Big(\int_{\mathbb{R}^n_v}\int_{\mathbb{R}^n_{v_*}}(\langle v'\rangle^{\alpha+(1-\delta)\gamma} + \langle v_*'\rangle^{\alpha+(1-\delta)\gamma})^2\big\|f(v_*')\big\|^2_{B^0_{p,1,x}}\big\|g(v')\big\|^2_{B^{\frac{n}{q}}_{q,1,x}} dv_* dv\Big)^{1/2} \\
= & C\Big(\int_{\mathbb{R}^n_{v'}}\int_{\mathbb{R}^n_{v_*'}}(\langle v'\rangle^{2(\alpha+(1-\delta)\gamma)} + \langle v_*'\rangle^{2(\alpha+(1-\delta)\gamma)})\big\|f(v_*')\big\|^2_{B^0_{p,1,x}}\big\|g(v')\big\|^2_{B^{\frac{n}{q}}_{q,1,x}} dv_*' dv'\Big)^{1/2} \\
= & C\bigg(\Big\|\big\|f(v_*')\big\|_{B^0_{p,1,x}}\Big\|_{L^2_\beta(\mathbb{R}_{v_*'})}\Big\|\big\|g(v')\big\|_{B^{\frac{n}{q}}_{q,1,x}}\Big\|_{L^2(\mathbb{R}_{v'})} \\
&\qquad + \Big\|\big\|f(v_*')\big\|_{B^0_{p,1,x}}\Big\|_{L^2(\mathbb{R}_{v_*'})}\Big\|\big\|g(v')\big\|_{B^{\frac{n}{q}}_{q,1,x}}\Big\|_{L^2_\beta(\mathbb{R}_{v'})}\bigg) \\
= & C\bigg(\big\|f(v_*')\big\|_{L^2_{\beta,v_*'}(B^0_{p,1,x})}\big\|g(v')\big\|_{L^2_{v'}(B^{\frac{n}{q}}_{q,1,x})} + \big\|f(v_*')\big\|_{L^2_{v_*'}(B^0_{p,1,x})}\big\|g(v')\big\|_{L^2_{\beta,v'}(B^{\frac{n}{q}}_{q,1,x})}\bigg),
\end{aligned}$$

where $\beta = \alpha + \gamma(1-\delta)$. For the second component of the nonlinear coupling Γ_2, we see the symmetry between f and g that

(6.6)
$$
\begin{aligned}
&\|\nu(v)^{-\delta}\Gamma_2(f,g)\|_{L^2_\alpha(\mathbb{R}^n_v;B^0_{p,1}(\mathbb{R}^n_x))}\\
&\quad=\Big\|\nu(v)^{-\delta}\int_{\mathbb{R}^n}\int_{\mathbb{S}^{n-1}} q(|v-v_*|,\omega)\sqrt{\mu(v_*)}f(v_*)g(v)d\omega dv_*\Big\|_{L^2_{\alpha,v}(B^0_{p,1,x})}\\
&\quad\le\Big\|\nu(v)^{-\delta}\int_{\mathbb{R}^n}\int_{\mathbb{S}^{n-1}} \big|q(|v-v_*|,\omega)\big|\sqrt{\mu(v_*)}\big\|f(v_*)g(v)\big\|_{B^0_{p,1,x}}d\omega dv_*\Big\|_{L^2_{\alpha,v}}\\
&\quad\le C\Big\|\nu(v)^{-\delta}\int_{\mathbb{R}^n}\int_{\mathbb{S}^{n-1}} \Big|q(|v-v_*|,\omega)\sqrt{\mu(v_*)}\Big|\big\|f(v_*)\big\|_{B^0_{p,1,x}}\big\|g(v)\big\|_{B^{\frac{n}{q}}_{q,1,x}}d\omega dv_*\Big\|_{L^2_{\alpha,v}}\\
&\quad\equiv I_2.
\end{aligned}
$$

Similar to the estimate for I_1 in (6.3) and (6.5), we obtain

(6.7)
$$
I_2 \le C\left(\big\|f(v_*)\big\|_{L^2_{\beta,v_*}(B^0_{p,1,x})}\big\|g(v)\big\|_{L^2_v(B^{\frac{n}{q}}_{q,1,x})}+\big\|f(v_*)\big\|_{L^2_{v_*}(B^0_{p,1,x})}\big\|g(v)\big\|_{L^2_{\beta,v}(B^{\frac{n}{q}}_{q,1,x})}\right).
$$

Now we assume that f_1 and f_2 are two mild weak solutions of the Boltzmann equation in the class $C\big([0,T);L^2_\beta(\mathbb{R}^n_v;B^0_{p,1}\cap B^{\frac{n}{q}}_{q,1}(\mathbb{R}^n_x))\big)$, then both of f_1 and f_2 are solutions of the integral equation (3.7), the difference of two solutions; $w(t)\equiv f_1(t)-f_2(t)$ also satisfies the following integral equation:

$$
\begin{aligned}
w(t)&=\int_0^t e^{-(t-s)\Lambda}\big(\Gamma(f_1(s),f_1(s))-\Gamma(f_2(s),f_2(s))\big)ds\\
&=\int_0^t e^{-(t-s)\Lambda}\big(\Gamma(f_1(s),w(s))+\Gamma(w(s),f_2(s))\big)ds\\
&=\int_0^t e^{-(t-s)\Lambda}\big(\Gamma_1(f_1(s),w(s))-\Gamma_2(f_1(s),w(s))\big)ds\\
&\quad+\int_0^t e^{-(t-s)\Lambda}\big(\Gamma_1(w(s),f_2(s))-\Gamma_2(w(s),f_2(s))\big)ds.
\end{aligned}
\tag{6.8}
$$

Then for any $1<r<\infty$ and $I=(0,T)$ with $T>0$, we see by (6.2)-(6.8) and Proposition 5.4 with $\delta=1$,

$$
\begin{aligned}
&\|w(t)\|_{L^r\big(I;L^2_{\alpha,v}(B^0_{p,1,x})\big)}\\
&\quad\le\Big\|\Big\|\int_0^t e^{-(t-s)\Lambda}\big(\Gamma_1(f_1(s),w(s))-\Gamma_2(f_1(s),w(s))\big)ds\Big\|_{L^2_{\alpha,v}(B^0_{p,1,x})}\Big\|_{L^r(I)}\\
&\qquad+\Big\|\Big\|\int_0^t e^{-(t-s)\Lambda}\big(\Gamma_1(w(s),f_2(s))-\Gamma_2(w(s),f_2(s))\big)ds\Big\|_{L^2_{\alpha,v}(B^0_{p,1,x})}\Big\|_{L^r(I)}\\
&\quad\equiv II_1+II_2.
\end{aligned}
\tag{6.9}
$$

(6.10)

$$
\begin{aligned}
II_1 \leq & C \Big\| \|\nu(v)^{-1}\Gamma_1(f_1(s),w(s))\|_{L^2_v(B^0_{p,1,x})} \\
& \qquad\qquad + \|\nu(v)^{-1}\Gamma_2(f_1(s),w(s))\|_{L^2_v(B^0_{p,1,x})} \Big\|_{L^r(I)} \\
\leq & C \Big\| \big\|f_1(v_*)\big\|_{L^2_{v_*}(B^{\frac{n}{q}}_{q,1,x})} \big\|w(v)\big\|_{L^2_{\beta,v}(B^0_{p,1,x})} \\
& \qquad\qquad + \big\|f_1(v_*)\big\|_{L^2_{\beta,v_*}(B^{\frac{n}{q}}_{q,1,x})} \big\|w(v)\big\|_{L^2_v(B^0_{p,1,x})} \Big\|_{L^r(I)} \\
& + C \Big\| \big\|f_1(v_*)\big\|_{L^2_{\beta,v_*}(B^{\frac{n}{q}}_{q,1,x})} \big\|w(v)\big\|_{L^2_v(B^0_{p,1,x})} \\
& \qquad\qquad + \big\|f_1(v_*)\big\|_{L^2_{v_*}(B^{\frac{n}{q}}_{q,1,x})} \big\|w(v)\big\|_{L^2_{\beta,v}(B^0_{p,1,x})} \Big\|_{L^r(I)} \\
\leq & CM\|w(t)\|_{L^r\left(I;L^2_{\beta,v}(B^0_{p,1,x})\right)}
\end{aligned}
$$

and

(6.11)

$$
\begin{aligned}
II_2 \leq & C \Big\| \|\nu(v)^{-1}\Gamma_1(w(s),f_2(s))\|_{L^2_v(B^0_{p,1,x})} \\
& \qquad\qquad + \|\nu(v)^{-1}\Gamma_2(w(s),f_2(s))\|_{L^2_v(B^0_{p,1,x})} \Big\|_{L^r(I)} \\
\leq & C \Big\| \big\|f_2(v)\big\|_{L^2_{v}(B^{\frac{n}{q}}_{q,1,x})} \big\|w(v_*)\big\|_{L^2_{\beta,v_*}(B^0_{p,1,x})} \\
& \qquad\qquad + \big\|f_2(v)\big\|_{L^2_{\beta,v}(B^{\frac{n}{q}}_{q,1,x})} \big\|w(v_*)\big\|_{L^2_{v_*}(B^0_{p,1,x})} \Big\|_{L^r(I)} \\
& + C \Big\| \big\|f_2(v)\big\|_{L^2_{\beta,v}(B^{\frac{n}{q}}_{q,1,x})} \big\|w(v_*)\big\|_{L^2_{v_*}(B^0_{p,1,x})} \\
& \qquad\qquad + \big\|f_2(v)\big\|_{L^2_{v}(B^{\frac{n}{q}}_{q,1,x})} \big\|w(v_*)\big\|_{L^2_{\beta,v_*}(B^0_{p,1,x})} \Big\|_{L^r(I)} \\
\leq & CM\|w(t)\|_{L^r\left(I;L^2_{\beta,v}(B^0_{p,1,x})\right)},
\end{aligned}
$$

where $\beta = \alpha + \gamma(1-\delta) = \alpha$ and

$$
M = \max_{i=1,2} \sup_{t \in I} \Big(\big\|f_i(t)\big\|_{L^2_v(B^{\frac{n}{q}}_{q,1,x})} + \big\|f_i(t)\big\|_{L^2_{\alpha,v}(B^{\frac{n}{q}}_{q,1,x})} \Big). \tag{6.12}
$$

Under the smallness assumption $M \leq 2M_0$, we obtain from (6.9)-(6.12) that

$$
\|w(t)\|_{L^r(I;L^2_v(B^0_{p,1,x}))} + \|w(t)\|_{L^r(I;L^2_{\alpha,v}(B^0_{p,1,x}))} \leq 0,
$$

which concludes that the solutions f_1 and f_2 coincides. This shows the proof of Theorem 6.1. ☐

Remark 5. The constant M appearing in (6.12) can be taken independent of T. Indeed, if we take, instead of (6.12) that

$$M = \max_{i=1,2} \sup_{t\in[0,\infty)} \left(\left\|f_i(t)\right\|_{L^2_v(B^{\frac{n}{q}}_{q,1,x})} + \left\|f_i(t)\right\|_{L^2_{\alpha,v}(B^{\frac{n}{q}}_{q,1,x})}\right), \tag{6.13}$$

then M does not depend on T. However if M is taken as (6.13), then we have to take M small such as $CM < 1$ in (6.10)-(6.11), and also M_0 in Corollary 6.2 small by $M \le 2M_0$.

Remark 6. We remark that the method of [3] and [2, Theorem 1.2] which treats non-cutoff case by Alexandre-Morimoto-Ukai-Xu-Yang is applicable to show a uniqueness of the original Boltzmann equation with angular cutoff condition such as $\mu^{1/4}g$ with $0 \le g \in L^\infty([0,T]\times\mathbb{R}^3_x; L^2_{\langle v\rangle^\alpha}(\mathbb{R}^3))$, $\alpha >> 1$.

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References

[1] Alexandre, R., Morimoto,Y., Ukai, S.,Xu,C.-J., Yang,T., *Local existence with mild regularity for the Boltzmann equation*, Kinet. Relat. Models **6** (4) (2013), 1011-1041.

[2] Alexandre, R., Morimoto, Y., Ukai, S., Xu, C.-J., Yang, T., *The Boltzmann equation without angular cutoff in the whole space: qualitative properties of solutions*, Arch. Ration. Mech. Anal. **202** (2011), 599-661.

[3] Alexandre, R., Morimoto, Y., Ukai, S., Xu, C.-J., Yang, T., *Uniqueness of solutions for the non-cutoff Boltzmann equation with soft potential.*, Kinet. Relat. Models. **4** (2011), 919-934.

[4] Alexandre, R., Villani, C., *On the Boltzmann equation for long-range interactions,* Comm. Pure Appl. Math. **55** (2002), 30-70.

[5] Amann, H., *Linear and Quasilinear Parabolic Problems. Vol I Abstract Linear Theory*, Monographs in Math. Vol **89**, Birkhäuser Verlag, Basel·Boston·Berlin, 1995.

[6] Bergh, J., Löfström, J., *Interpolation Spaces; an introduction*, Springer-Verlag, Berlin, 1976.

[7] Chemin, J.-Y., Lerner, N., *Flot de champs de vecteurs non lipschitziens et équations de Navier-Stokes*, J. Differential Equation. **121** (1995), 314-328.

[8] Desvillettes, L., Villani, C., *On the spatially homogeneous Landau equation for hard potentials, I.. Existence, uniqueness and smoothness,* Comm. Partial Differential Equations, **25** (2000), 179-259.

[9] Desvillettes, L., Villani, C., *On the trend to global equilibrium for spatially inhomogeneous kinetic systems: the Boltzmann equation,* Invent. Math., **159** (2005), 245-316.

[10] DiPerna, R.J., Lions, P.-L., *On the Cauchy problem for Boltzmann equations: global existence and weak stability*, Ann. Math. **130** (2) (1989), 321-366.

[11] Duan, R.-J.,*On the Cauchy problem for the Boltzmann equation in the whole space: global existence and uniform stability in $L^2_\xi(H^N_x)$*, J. Differential. Equation **244** (12) (2008), 3204-3234.

[12] Duan, R., Liu, S., Xu, J., *Global wellposedness in spatially critical Besov space for Boltzmann equation*, Arch. Rat. Mech. Anal. **55** (2015), published on line.

[13] Grad, H., *Asymptotic Theory of Boltzmann Equation, Rarefied Gas Dynamitcs 1*, Acadmic Press, 1963.

[14] Gressman, P.T., Strain, R.M., *Global classical solutions of the Boltzmann equation without angular cut-off*, J. Am. Math. Soc. **24** (2011), 771-847.

[15] Guo, Y., *The Landau equation in a periodic box*, Commun. Math. Phys. **231** (2002), 391-434.

[16] Guo, Y., *Classical solutions to the Boltzmann equation for molecules with an angular cutoff*, Arch. Ration. Mech. Anal. **169**(4) (2003), 305-353.

[17] Guo, Y.,*The Boltzmann equation in the whole space*, Indiana Univ. Math. J. **53** (2004), 1081-1094.

[18] Kawashima, S., *The Boltzmann equation and thirteen moments*, Japan. J. Appl. Math. **7** (1990), 301-320.

[19] Lions, P.L., *Regularity and compactness for Boltzmann collision operator without angular cut-off*, C.R.Acad. Sci. Paris, Ser. 1, **326** (1998), 37-41.

[20] Liu, T.-P., Yang, T., Yu, S.-H., *Energy method for the Boltzmann equation*, Phys. D **188** (2004), 178-192.

[21] Liu, T.-P., Yu, S.-H., *Boltzmann equation: micro-macro decompositions and positivity of shock profiles*, Commun. Math. Phys. **246** (1) (2004), 133-179.

[22] Morimoto, Y., Sakamoto, S, *Global solutions in the critical Besov space for the non-cutoff Boltzmann equation*, J. Differential Equations, **261** (2016), 4073-4134.

[23] Nishida, T., Imai., K., *Global solutions to the initial value problem for the nonlinear Boltzmann equation*, Publ. Res. Inst. Math. Sci., Kyoto Univ., **12** (1977), 229-239.

[24] Ogawa, T., *Nonlinear Evolutionary Partial Differential Equations –Method of Real and Harmonic Analysis-*, Springer-Verlag, to appear.

[25] Ogawa, T., Shimizu, S., *End-point maximal L^1-regularity for the Cauchy problem to a parabolic equation with variable coefficients*, Math. Ann. **365** no.1 (2016), 661-705.

[26] Prüss, J., Simonett, G., *Moving Interfaces and Quasi-linear Parabolic Differential Equations*, Monographs in Math. **105**, Birkhäuser, Basel, 2016.

[27] Shizuta, Y., Asano, K., *Global solutions of the nonlinear Boltzmann equation in a bounded domain*, Proc. Japan Acad., **53**, (1977), 3-5.

[28] Triebel, H., *Theory of Function Spaces,* Birkhäuser, Basel, 1983.

[29] Ukai, S., *On the existence of global solutions of mixed problem for non-linear Boltzmann equation*. Proc. Japan Acad. **50** (1974), 179-184.

[30] Ukai, S., *Les solutions globales de l'équation de Boltzmann dans l' espace tout entier et dans le demi-espace*, C. R. Acad. Sci. Paris **282A** (1976) 317-320.

[31] Ukai, S., *Solutions of the Boltzmann equation*, Studies of Mathematics and Its Applications **18** (Eds. Mimura M. and Nishida T.), Kinokuniya/North-Holland, Tokyo, 37-96, 1986.

[32] Ukai, S., *Research on the Boltzmann equations–Past and Future–*, Mathematical Society of Japan, Annual Meeting Plenary Talk Slide (in Japanese), 2010, Math. Soc. Japan.

[33] Ukai, S., Asano, K., *On the Cauchy problem of the Boltzmann equation with a soft potential*, Publ. Res. Inst. Math. Sci. **18** (2) (1982), 477-519.

[34] Ukai, S., Yang, T., *The Boltzmann equation in the space $L^2 \cap L^\infty_\beta$: Global and time periodic solutions*, Anal. Appl. **4** (3) (2006), 263-310.

[35] Villani, C., *On a new class of weak solutions to the spatially homogeneous Boltzmann and Landau equations*, Arch. Rat. Mech. Anal. **143** (1998), 273-307.